1. A uniform lamina *ABC* of mass *m* is in the shape of an isosceles triangle with *AB* = *AC* = 5*a* and *BC* = 8*a*.

(a) Show, using integration, that the moment of inertia of the lamina about an axis through *A*, parallel to *BC*, is $\frac{9}{2}ma^2$.

(6)

The foot of the perpendicular from *A* to *BC* is *D*. The lamina is free to rotate in a vertical plane about a fixed smooth horizontal axis which passes through *D* and is perpendicular to the plane of the lamina. The lamina is released from rest when *DA* makes an angle α with the downward vertical. It is given that the moment of inertia of the lamina about an axis through *A*, perpendicular to *BC* and in the plane of the lamina, is $\frac{8}{3}ma^2$.

(b) Find the angular acceleration of the lamina when *DA* makes an angle θ with the downward vertical.

(8)

Given that α is small,

(c) fin d an approximate value for the period of oscillation of the lamina about the vertical.

(2)
(Total 16 marks)

2. A pendulum consists of a uniform rod *AB*, of length 4*a* and mass 2*m* , whose end *A* is rigidly attached to the centre *O* of a uniform square lamina *PQRS*, of mass 4*m* and side *a*. The rod *AB* is perpendicular to the plane of the lamina. The pendulum is free to rotate about a fixed smooth horizontal axis *L* which passes through *B*. The axis *L* is perpendicular to *AB* and parallel to the edge *PQ* of the square.

(a) Show that the moment of inertia of the pendulum about *L* is $75ma^2$.

(4)

The pendulum is released from rest when BA makes an angle α with the downward vertical through B, where $\tan \alpha = \frac{7}{24}$. When BA makes an angle θ with the downward vertical through B, the magnitude of the component, in the direction AB, of the force exerted by the axis L on the pendulum is X.

(b) Find an expression for X in terms of m, g and θ.

(9)

Using the approximation $\theta \approx \sin \theta$,

(c) find an estimate of the time for the pendulum to rotate through an angle α from its initial rest position.

(6)
(Total 19 marks)

3. A uniform circular disc, of mass m, radius a and centre O, is free to rotate in a vertical plane about a fixed smooth horizontal axis. The axis passes through the mid-point A of a radius of the disc.

(a) Find an equation of motion for the disc when the line AO makes an angle θ with the downward vertical through A.

(5)

(b) Hence find the period of small oscillations of the disc about its position of stable equilibrium.

(2)

When the line AO makes an angle θ with the downward vertical through A, the force acting on the disc at A is **F**.

(c) Find the magnitude of the component of **F** perpendicular to AO.

(5)
(Total 12 marks)

4. A uniform rod AB, of mass m and length $2a$, is free to rotate in a vertical plane about a fixed smooth horizontal axis through A and perpendicular to the plane. The rod hangs in equilibrium with B below A. The rod is rotated through a small angle and released from rest at time $t = 0$.

(a) Show that the motion of the rod is approximately simple harmonic.

(4)

(b) Using this approximation, find the time t when the rod is first vertical after being released.

(2)
(Total 6 marks)

5. A uniform square lamina $ABCD$, of mass m and side $2a$, is free to rotate in a vertical plane about a fixed smooth horizontal axis L which passes through A and is perpendicular to the plane of the lamina. The moment of inertia of the lamina about L is $\dfrac{8ma^2}{3}$.

Given that the lamina is released from rest when the line AC makes an angle of $\dfrac{\pi}{3}$ with the downward vertical,

(a) find the magnitude of the vertical component of the force acting on the lamina at A when the line AC is vertical.

(7)

Given instead that the lamina now makes small oscillations about its position of stable equilibrium,

(b) find the period of these oscillations.

(5)
(Total 12 marks)

6. A uniform circular disc, of mass m and radius r, has a diameter AB. The point C on AB is such that $AC = \frac{1}{2}r$. The disc can rotate freely in a vertical plane about a horizontal axis through C, perpendicular to the plane of the disc. The disc makes small oscillations in a vertical plane about the position of equilibrium in which B is below A.

(a) Show that the motion is approximately simple harmonic.

(6)

(b) Show that the period of this approximate simple harmonic motion is $\pi\sqrt{\left(\dfrac{6r}{g}\right)}$.

(1)

The speed of *B* when it is vertically below *A* is $\sqrt{\left(\frac{gr}{54}\right)}$. The disc comes to rest when *CB* makes an angle α with the downward vertical.

(c) Find an approximate value of α.

(3)
(Total 10 marks)

7. (a) Prove, using integration, that the moment of inertia of a uniform circular disc, of mass *m* and radius *a*, about an axis through its centre *O* perpendicular to the plane of the disc is $\frac{1}{2}ma^2$.

(4)

The line *AB* is a diameter of the disc and *P* is the mid-point of *OA*. The disc is free to rotate about a fixed smooth horizontal axis *L*. The axis lies in the plane of the disc, passes through *P* and is perpendicular to *OA*. A particle of mass *m* is attached to the disc at *A* and a particle of mass 2*m* is attached to the disc at *B*.

(b) Show that the moment of inertia of the loaded disc about *L* is $\frac{21}{4}ma^2$.

(6)

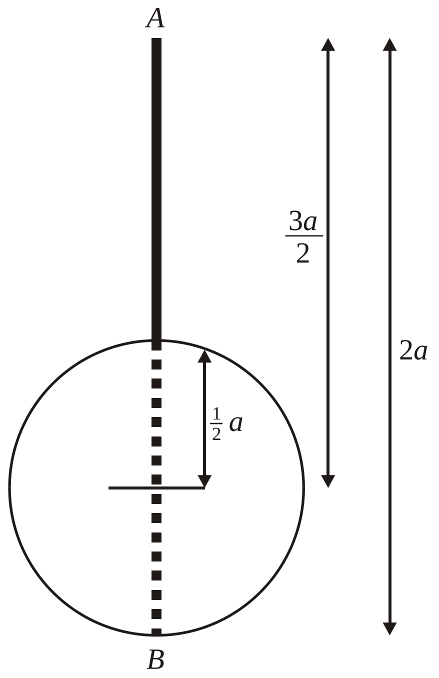
At time $t = 0$, *PB* makes a small angle with the downward vertical through *P* and the loaded disc is released from rest. By obtaining an equation of motion for the disc and using a suitable approximation,

(c) find the time when the loaded disc first comes to instantaneous rest.

(8)
(Total 18 marks)

8. (a) Show by integration that the moment of inertia of a uniform disc, of mass m and radius a, about an axis through the centre of disc and perpendicular to the plane of the disc is $\frac{1}{2}ma^2$.

(3)



A uniform rod AB has mass $3m$ and length $2a$. A uniform disc, of mass $4m$ and radius $\frac{1}{2}a$, is attached to the rod with the centre of the disc lying on the rod a distance $\frac{3}{2}a$ from A. The rod lies in the plane of the disc, as shown in the diagram above. The disc and rod together form a pendulum which is free to rotate about a fixed smooth horizontal axis L which passes through A and is perpendicular to the plane of the pendulum.

(b) Show that the moment of inertia of the pendulum about L is $\frac{27}{2}ma^2$.

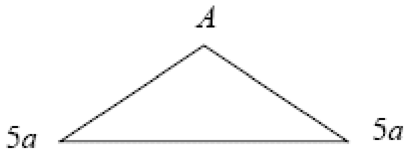
(3)

The pendulum makes small oscillations about its position of stable equilibrium.

(c) Show that the motion of the pendulum is approximately simple harmonic, and find the period of the oscillations.

(6)
(Total 12 marks)

1. (a)



$$\delta A = \frac{8x}{3}\delta x$$ M1 A1

$$\delta m = \frac{8x}{3}\delta x.\frac{m}{12a^2} \quad \text{or} \quad \delta m = \frac{8x}{3}\delta x.\rho$$ DM1

$$\delta I = \frac{8x}{3}\delta x.\frac{m}{12a^2}x^2 \quad (= \frac{2m}{9a^2}x^3\delta x)$$ A1

$$I = \int_0^{3a} \frac{2m}{9a^2}x^3 \ \mathrm{d}x$$ M1

$$= \frac{2m}{9a^2}\left[\frac{x^4}{4}\right]_0^{3a}$$

$$= \frac{9ma^2}{2}*$$ A1 6

(b)

$$I_A = \frac{9ma^2}{2} + \frac{8ma^2}{3} = \frac{43ma^2}{6} \quad \text{(perp axes rule)}$$ M1 A1

$$I_A = I_G + m(2a)^2 \quad \text{(parallel axes rule)}$$ DM1 A1

$$I_D = I_G + ma^2 \quad \text{(parallel axes rule)}$$ A1

$$I_D = \frac{43ma^2}{6} - 3ma^2 = \frac{25ma^2}{6}$$ A1

$$mga\sin\theta = -\frac{25ma^2}{6}\ddot{\theta}$$ M1

$$\ddot{\theta} = -\frac{6g}{25a}\sin\theta$$ A1 8

(c) For small θ, $\ddot{\theta} = -\frac{6g}{25a}\theta$ SHM M1

$$T = 2\pi\sqrt{\frac{25a}{6g}} = 5\pi\sqrt{\frac{2a}{3g}}$$ A1 2

[16]

2. (a) $\frac{1}{3}2m(4a)^2 + \frac{1}{12}4ma^2 + 4m(4a)^2$ B1 M1 A1

$= \frac{32}{3}ma^2 + \frac{1}{3}ma^2 + 64ma^2$

$= 75ma^2$ * A1 4

(b) $\frac{1}{2}\ 75ma^2\omega^2 = 2mg2a(\cos\theta - \cos\alpha) + 4mg4a(\cos\theta - \cos\alpha)$ M1 A2

$a\omega^2 = \frac{8}{15}g(\cos\theta - \frac{24}{25}) = \frac{8}{375}g(25\cos\theta - 24)$ A1

$X - 6mg\cos\theta = 2m2a\omega^2 + 4m4a\omega^2 = 20ma\omega^2$ M1 A2

$X = 6mg\cos\theta + 20m\frac{8}{375}g(25\cos\theta - 24)$ D M1

$= \frac{50mg\cos\theta}{3} - \frac{256mg}{25}$ A1 9

(c) $-2mg\ 2a\sin\theta - 4mg\ 4a\sin\theta = 75ma^2\ddot{\theta}$ M1 A1

$\ddot{\theta} = -\frac{4g}{15a}\sin\theta$ A1

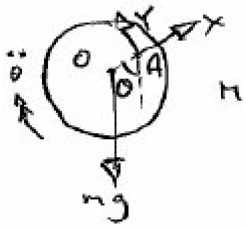
$\approx -\frac{4g}{15a}\theta$, SHM M1

Time = $\frac{1}{4}2\pi\sqrt{\frac{15a}{4g}}$ M1

$\frac{\pi}{4}\sqrt{\frac{15a}{g}}$ A1 6

[19]

3. (a)



$I_A = \frac{1}{2}mc^2 + m\left(\frac{1}{2}a^2\right) = \frac{3}{4}mc^2$ M1 A1

M(A1, $-mg\frac{a}{2}\sin\theta = \frac{3}{4}ma^2\ddot{\theta}$ M1

$-\frac{2g}{3a}\sin\theta = \ddot{\theta}$ A2 5

(b) For small θ, $-\frac{2g}{3a}\theta = \ddot{\theta}$ M1

$\text{T} = 2\pi\sqrt{\frac{3a}{2g}}$ A1 2

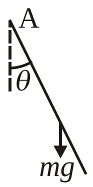
(c) R(↖), $\text{Y} - mg\sin\theta = m\frac{a}{2}\ddot{\theta}$ M1 A2

$\Rightarrow \text{Y} = mg\sin\theta + \frac{ma}{2}\left(-\frac{2g}{3a}\sin\theta\right)$ M1

$= \frac{2mg\sin\theta}{3}$ A1 5

[12]

4.



(a) m(A) : $\frac{4}{3}ma^2\ddot{\theta} = -\,mga\sin\theta$ M1 A1

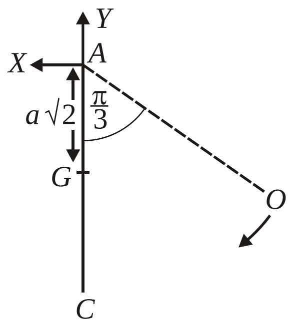
Small $\theta \Rightarrow \sin\theta \simeq \theta \rightarrow\ \ddot{\theta} = -\frac{3g}{4a}\theta$ M1

$\therefore$ Approx. SHM A1 4

(b) Time = $\frac{1}{4}$ period = $\frac{1}{4}\ .\ 2\pi\sqrt{\frac{4a}{3g}} = \pi\sqrt{\frac{a}{3g}}$ M1 A1 2

[6]

5. (a)



Energy: $\frac{1}{2}.\frac{8ma^2}{3}\dot{\theta}^2 = mga\sqrt{2}\left(1-\cos\frac{\pi}{3}\right)$ M1 A1

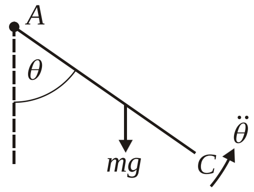
$\Rightarrow\ \dot{\theta}^2 = \frac{3g\sqrt{2}}{8a}$ A1

$R\uparrow$: $\gamma - mg = ma\sqrt{2}\dot{\theta}^2$ M1 A1

$= ma\sqrt{2}.\frac{3g\sqrt{2}}{8a}$ M1

$\Rightarrow \underline{\gamma = \frac{7mg}{4}}$ A1 7

(b)



M(*L*): $-mga\sqrt{2}\sin\theta = \frac{8}{3}ma^2\ddot{\theta}$ M1 A1 A1

For small $\theta \sin\theta \cong \theta$

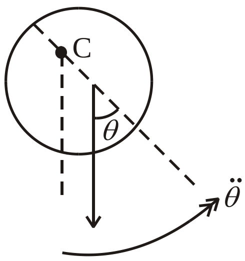
$\Rightarrow -\frac{3g\sqrt{2}}{8a}\dot{\theta} \cong \ddot{\theta}$ M1

Approx SHM

$\Rightarrow$ period $= 2\pi\sqrt{\frac{8a}{3g\sqrt{2}}}$ A1 5

[12]

6. (a)



$I_C = \frac{1}{2}mr^2 + m(\frac{1}{2}r)^2 = \frac{3}{4}mr^2$ M1 A1

M(c): $\frac{3}{4}mr^2\ddot{\theta} = -mg\frac{r}{2}\sin\theta$ M1 A1 ft

$\sin\theta \approx \theta \Rightarrow \ddot{\theta} \approx -\frac{2g}{3r}\theta$ approx. SHM M1 A1 6

(b) Period $= 2\pi\sqrt{\frac{3r}{2g}} = \pi\sqrt{\frac{6r}{g}}$ (*) A1 1

(c) $\dot{\theta}_{\max} = \omega\alpha \Rightarrow \frac{2}{3r}\sqrt{\frac{gr}{54}} = \sqrt{\frac{2g}{3r}}\,\alpha$ M1 A1

$\Rightarrow \alpha = \frac{1}{9}^{c}$ A1 3

[10]

<u>Or</u>

$mg \times \frac{1}{2}r(1 - \cos\alpha) = \frac{1}{2}(\frac{3}{4}mr^2)\left(\frac{gr}{54}\right)\left(\frac{2}{3r}\right)^2$ M1 A1

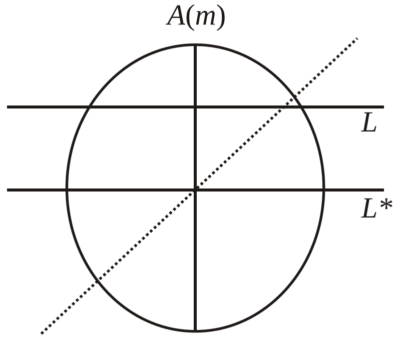
$\cos\alpha = \frac{161}{162}$, $\alpha = 6.4°$ (AWRT) or 0.11^c (AWRT) A1

7. (a) $(\delta I) = (\rho)2\pi r\delta r.r^2$ M1

Using $(\rho) = \frac{m}{\pi a^2}$ M1

Completion: $I = \frac{2m}{a^2}\left[\frac{r^4}{4}\right]_0^a = \frac{1}{2}ma^2$ (*) M1 A1 4

(b)



Disc: Use of $\perp^r$ axis theorem to find I_{L*} M1

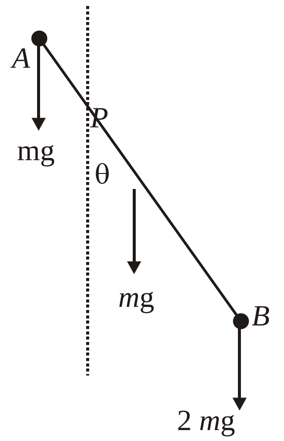
$I_{L*} = \frac{1}{2}(\frac{1}{2}ma^2) = \frac{1}{4}ma^2$ A1

Use of parallel axis theorem

$I_L = \frac{1}{4}ma^2 + m\left(\frac{a}{2}\right)^2 = \frac{1}{2}ma^2$ M1 A1

For loaded disc: $I = \frac{1}{2}ma^2 + m\left(\frac{a}{2}\right)^2 + 2m\left(\frac{3a}{2}\right)^2 = \frac{21}{4}ma^2$ (*) M1 A1 cso 6

(c)



$I\ddot{\theta} = \left\{ mg\left(\frac{a}{2}\right)\sin\theta - mg\left(\frac{a}{2}\right)\sin\theta - 2mg\left(\frac{3a}{2}\right)\sin\theta \right\}$ M1 A1 A1

[A1 for signs, A1 "terms"]

$\left[\frac{21}{4}ma^2\ddot{\theta} = -3mga\sin\theta\right]$

For small angles $\theta \approx \sin\theta \Rightarrow \frac{21}{4}ma^2\ddot{\theta} = -3mga\theta$ M1

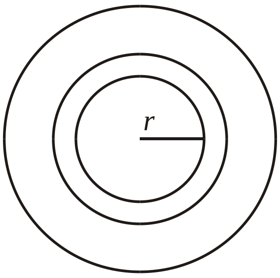
$\ddot{\theta} = -\frac{4g}{7a}\theta$ A1 ft

$\Rightarrow$ SHM with $\omega^2 = \frac{4g}{7a}$ M1

Time = $\frac{\pi}{\omega}$; $= \pi\sqrt{\frac{7a}{4g}}$ or $\frac{\pi}{2}\sqrt{\frac{7a}{g}}$ M1; A1 8

[18]

8. (a)



MI of element = $2\pi\rho r\delta r \times r^2$ M1

$m = \pi\rho a^2$

$\Rightarrow I = \dfrac{2m}{a^2}\displaystyle\int_0^a r^3\,dr$ M1

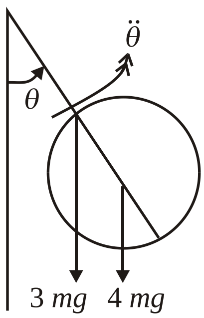
$= \dfrac{2m}{a^2}\left[\dfrac{r^4}{4}\right]_0^a = \tfrac{1}{2}ma^2$ A1 3

(b) $I = I_{rod} + I_{disc} = \tfrac{4}{3} \times 3m \times a^2, + \tfrac{1}{2} \times 4m\left(\dfrac{a}{2}\right)^2 + 4m\left(\dfrac{3a}{2}\right)^2$ B1, M1

$= 4ma^2 + \dfrac{ma^2}{2} + 9ma^2$

$= \tfrac{27}{2}ma^2$ A1 3

(c)



$\tfrac{27}{2}ma^2\,\ddot{\theta} = -3mg \times a\sin\theta - 4mg \times \dfrac{3a}{2}\sin\theta$ M1 A2, 1, 0

$= -9mga\sin\theta$

$\ddot{\theta} = -\dfrac{2g}{3a}\sin\theta$

Small oscillations $\Rightarrow \sin\theta \approx \theta$	M1	
$\Rightarrow \ddot{\theta} = -\frac{2g}{3a}\theta$ which is SHM	A1	
$T = 2\pi\sqrt{\frac{3a}{2g}}$	A1	6

[12]

1. This question proved to be too difficult for many. Few completely correct solutions were seen.

Part (a) was very poorly answered. In order to find the mass of a strip, the ratio of base to height of the triangle was required. The height was easily found using a 3, 4,5 triangle yet a number of candidates made errors here with $\frac{3}{5}$ rather than $\frac{4}{3}$ seen quite often. Many candidates were unable to use appropriate methods for calculating the moment of inertia of a strip about the required axis through *A*. Many tried to use an axis through *BC* instead. A number used an incorrect density, failing to understand that *m* was the mass for the whole triangle and not half of it. Methods used were often very difficult to follow. There was often little indication of what the candidate was trying to do.

For part (b), very many thought that $\frac{8}{3}ma^2$ was the moment of inertia required in their solution. Very few realised that they needed to use this, together with the answer from part (a), and the perpendicular axes rule. Of those who did, few appreciated that they also needed to use the parallel axes rule, in order to find the moment of inertia about the axis required. Using their moment of inertia, in an equation of motion, to find the angular acceleration also proved to be a stumbling block for many, with 2*a* rather than *a* often seen. A significant number chose, instead, to differentiate an energy equation and both methods were generally used correctly by those who got this far.

In part (c) many candidates failed to write down the actual SHM equation for $\ddot{\theta}$ in terms of θ before writing down the periodic time, thereby losing all the marks.

2. The weakness of candidates when answering questions about rotational mechanics was shown by the large number who only attempted part (a) of this question. This sometimes involved blatant “fudging”.

In part (b), the attempt at the energy equation was often reasonable, but using Newton's second law along the rod often contained dimensional errors, such as multiplying the component of the weight by a distance.

In the final part it was necessary to obtain an expression for the angular acceleration either by taking moments or by differentiating the energy equation from part (b). Without this starting point, it was not possible to use the approximation $\theta \approx \sin\theta$ and so few marks could be obtained. There are still a number of candidates who learn a formula for the period of a compound pendulum and this should be strongly discouraged as the questions are usually designed to test understanding and the ability to work from 1st principles.

3. There were many good solutions to part (a) but a missing minus sign was a common error. Some started off with an energy equation and then differentiated. Some (mostly, but not all, from overseas) appeared not to understand the instruction about finding an equation of motion for the disc. Many candidates followed the instruction to use their answer to part (a) to answer part (b). In fact many had already replaced sin θ with θ in part (a) which could cause problems in part (c). Some however insisted on using the formula they had learned for the period of an approximate SHM and scored nothing in part (b). Many candidates made a reasonable attempt at the final part, using Newton's second law. Errors arose from signs, using θ instead of sin θ or finding the wrong component.

4. Weaker candidates failed to write down an equation of rotational motion in part (a) and hence could make little progress here. More able candidates however had little difficulty with this question and scored full marks easily.

5. Better candidates did very well in this question but others had a general idea of the methods but could not implement them correctly. The most common errors were: in (a), not using the motion of the centre of mass and using $Y - mg\cos(\pi/3)$ instead of $Y - mg$. and in (b), omitting the negative sign in the equation of motion. Candidates should not quote the formula for the period of oscillation – they should always work from first principles.

6. Parts (a) and (b) were generally well done. A few omitted the negative sign in the equation of motion but generally this was very well done. Part (c) proved to be very demanding and very few correct answers were seen. Many confused linear and angular quantities and, even among those who realised that they had to deal with the angular speed, went wrong in dealing with the distance from the axis.

7. Part (a) often provided candidates with 4 marks, although some candidates did need to “fudge” a little, and the approach of some candidates did involve a considerable amount of work. In part (b) good candidates did set out their working very clearly but often it was difficult to know what candidates were doing. A very frequent “solution”, with no explanation, was

I = $\frac{1}{2}ma^2 + m\left(\frac{a}{2}\right)^2 + m\left(\frac{a}{2}\right)^2 + 2m\left(\frac{3a}{2}\right)^2$ followed by = $\frac{21}{4}ma^2$, the given answer. For these candidates, who had not considered the need to apply the perpendicular axis theorem, the answer should have been $\frac{22}{4}ma^2$!

In part (c) it was quite common for candidates to omit one of the particles or make a sign error in the equation of motion. If the resulting equation, after the approximation of θ for $\sin\theta$, did not represent SHM the final three marks were not available. As stated in the introduction, if

$T = 2\pi\sqrt{\frac{I}{mgh}}$ was used, without proof, then only 3 marks were available for a correct answer.

8. No Report available for this question.